A simple short derivation of the Clebsch-Gordan coefficients

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Abstract

We give a simple short derivation of the Clebsch-Gordan coefficients, motivated by simple introductory facts of quantum angular momentum. The derivation uses only binomial coefficients, eschewing raising- and lowering-operators, group theory, spinors and calculus.

Keywords: Quantum angular momentum coupling, Clebsch-Gordan coefficients.

Resumen

Damos una derivación sencilla de los coeficientes de Clebsch-Gordan, motivada por temas introductorios de momento angular cuántico. La derivación sólo utiliza coeficientes binomiales, evitando los operadores para subir o bajar índices, teoría de grupos, espinores y cálculo.

Palabras clave: Acoplamiento de momento angular cuántico, coeficientes de Clebsch-Gordan.

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I. INTRODUCTION

All courses on quantum mechanics include angular momentum, which is essential for understanding (say) atoms and molecules: how protons and electrons combine as atoms, and how atoms combine as molecules. In an atom the angular momentum of its many electrons add (combine, couple) to give the total angular momentum j of the atom.

Thus we wish to know how j_1 and j_2 add to give $j : j_1 + j_2 = j$. The answer is given by the Clebsch-Gordan coefficients (CGCs) $(j_1 j_2 m_1 m_2 \mid jm)$ which relate the angular momentum state $\mid jm)$ to states $\mid j_1 m_1)$ and $\mid j_2 m_2)$:

$$\mid jm) = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \delta(m_1 + m_2, m)(j_1 j_2 m_1 m_2 \mid jm) \mid j_1 m_1) \mid j_1 m_2). \quad (1)$$

We shall derive the CGCs, assuming the following elementary facts of $\mid jm)$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2} \ldots$, $m = -j..+j$, see §3:

- Any $\mid jm)$ can be written in terms of $j = \frac{1}{2}$ as

$$\mid jm) = \frac{\alpha^{j+m} \beta^{j-m}}{\sqrt{(j+m)!(j-m)!}}, \quad (2)$$

where $\alpha = \mid +) \sim \mid j = \frac{1}{2}, m = +\frac{1}{2})$ and $\beta = \mid -) \sim \mid j = \frac{1}{2}, m = -\frac{1}{2})$; this is also the intrinsic angular momentum (spin $s = \frac{1}{2}$) of an electron.

- For fixed j_1 and j_2 we have $j_1 + j_2 \le j \le \mid j_1 - j_2 \mid$ (triangle rule), so the states $\mid 00) \mid jm)$ and $\mid jm)$ have the same j and m.

The state $\gamma = \alpha_1 \beta_2 - \alpha_2 \beta_1$ is $\mid 00)$, an *invariant*; $j = 0$ shows the zero-angular-momentum and no-preferred-direction of spherical symmetry. Thus γ^q is also an invariant: $\mid jm)$ and $\gamma^q \mid jm)$ have the same j and m. We say that they transform $(\sim)$ in the same way (under rotation), $\mid jm) \sim \gamma^q \mid jm)$.

- We assume $\alpha \sim \alpha_1 + \alpha_2$ and $\beta \sim \beta_1 + \beta_2$ and note that $j = j_1 + j_2$.

Very few text-books on quantum mechanics derive CGCs: some quote the result (13) without proof, and most go no further (if that) than quoting (1). The very few derivations often use the `advanced' unfamiliar mathematics of group theory and spinors. Hence we offer the following simple short derivation, *c.f.* [1].

II. DERIVATION

For fixed j_1 and j_2 we will derive the CGCs of the allowed j. Then §1 implies that

$$f(j,m)=\left[(\alpha_1\beta_2-\alpha_2\beta_1)^q\right]\left[\frac{(\alpha_1+\alpha_2)^{j+m}(\beta_1+\beta_2)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\right]\sim|\,jm). \quad (3)$$

Expanding (3) by the binomial theorem gives

$$f(j,m)=\sum_{rpt}(-1)^r\binom{q}{r}\binom{j+m}{p}\binom{j-m}{t}[\alpha_1^{p+r}\beta_1^{q-r+t}][\alpha_2^{q-r-p+j+m}\beta_2^{r-t+j-m}] \quad (4)$$

where $\binom{a}{b}=a!/b!(a-b)!$ is a binomial coefficient. Comparison of the $[\ldots]_1$ and $[\ldots]_2$ of (4) to (2) gives the substitutions

$$j_1+m_1=p+r\,,\quad j_1-m_1=q-r+t\,,$$
$$j_2+m_2=q-r-p+j+m,\quad j_2-m_2=r-t+j-m, \quad (5)$$

which we manipulate to give $\sum_{m_1m_2r}$ by

$$p=j_1+m_1-r,\ \ t=r-m+j-j_2+m_2\,,\ \ q=j_1+j_2-j. \quad (6)$$

Substituting (6) and (2) into (4) and simplifying (expand binomial coefficients and cancel factorials) then gives

$$f(j,m)=\sum_{m_1m_2}|j_1m_1)\,|\,j_2m_2)(-1)^{j_1+j_2-j}\,\delta(m_1+m_2,m)$$
$$\sum_r\frac{(-1)^r(j_1+j_2-j)!\sqrt{(j+m)!(j-m)!(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}}{r!(j_1+j_2-j-r)!(j_1+m_1-r)!(j-j_1+m_2+r)!(j-m_1-j_2+r)!(j_2-m_2-r)!}. \quad (7)$$

Comparing (1) with (7) shows that the CGCs are effectively the $\sum_r$ but we must ensure that $f(j,m)$ is normalised before we can call it $|\,jm)$; this means that, with $m_2=m-m_1$ we must have

$$N^2\sum_{m_1}(j_1j_2m_1m-m_1\,|\,jm)^2=1, \quad (8)$$

from which we calculate the normalisation constant N. Since N is independent of m, we choose the special case $m=j$ and thus have $(j_1j_2m_1j-m_1\,|\,jj)$ which, from (7), has the single term $r=j_2-m_2$ in $\sum_r$; simplifying then gives

$$(j_1j_2m_1j-m_1\,|\,jj)=N(-1)^{j_2-j+m_1}\sqrt{\frac{(j_1+m_1)!(j_1-m_1)!(j_2+m-m_1)!(j_2-m+m_1)!}{(2j)!}}$$
$$\binom{j_1+j_2-j}{j_2-j+m_1}\binom{2j}{j_1-j_2+j}, \quad (9)$$

Substituting (9) into (8) and simplifying gives

$$N^2\binom{2j}{j_1-j_2+j}^2\frac{(j_1+j_2-j)!^2}{(2j)!}\sum_{m_1}\frac{(j_1+m_1)!(j_2+j-m_1!)}{(j_2-j+m_1)!(j_1-m_1)!}=1. \quad (10)$$

We now state that direct use of the properties of the binomial coefficients (no physics here) gives the $\sum_{m_1}$ as [2],

$$\sum_{m_1}\frac{(j_1+m_1)!(j_2+j-m_1!)}{(j_2-j+m_1)!(j_1-m_1)!}=(j_2-j_1+j)!(j_1-j_2+j)!\binom{j_1+j_2+j+1}{j_1+j_2-j}, \quad (11)$$

so, with (11) in (10),

$$N=\sqrt{\frac{(2j+1)(j_1-j_2+j)!(j_2-j_1+j)!}{(j_2+j_1-j)!(j_1+j_2+j+1)!}}. \quad (12)$$

We now insert N of (12) into (8) and (7) and simplify, finally giving [2]

$$(j_1j_2m_1m_2\,|\,jm)=\delta(m_1+m_2,m)\sqrt{\frac{(2j+1)(j_1-j_2+j)!(j_2-j_1+j)!(j_1+j_2-j)!}{(j_1+j_2+j+1)!}}$$
$$\sum_r\frac{(-1)^{r+j_1+j_2-j}\sqrt{(j+m)!(j-m)!(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}}{r!(j_1+j_2-j-r)!(j_1+m_1-r)!(j-j_1+m_2+r)!(j-m_1-j_2+r)!(j_2-m_2-r)!}. \quad (13)$$

III. CONCLUDING REMARKS

We believe (hope) that our simple short derivation of the Clebsch-Gordan coefficients will be accessible to someone familiar only with some introductory facts of quantum angular momentum. The derivation uses only binomial coefficients: it avoids raising and lowering-operators, group theory, spinors and calculus. It also automatically gives $q=j_1+j_2-j$ in (3), and uses the simplest invariant $\gamma=(\alpha_1\beta_2-\alpha_2\beta_1)$ -the familiar *singlet* state of a two-electron system.

Define the following operators [3, 2]:

$$j_+ = \alpha \frac{\partial}{\partial \beta}, \quad j_- = \beta \frac{\partial}{\partial \alpha}, \quad j_z = \alpha \frac{\partial}{\partial \alpha} - \beta \frac{\partial}{\partial \beta},$$
$$j_\pm = j_x \pm i j_y \ , \qquad (14)$$

and apply $\mathbf{j} = \mathbf{j}(j_+, j_-, j_z)$ to (2) by simple differentiation:

$$j_\pm \mid jm) = \sqrt{(j \mp m)(j \pm m + 1)} \mid jm \pm 1)$$
$$j_z \left| jm \right) = m \left| jm \right)$$
$$\mathbf{j}^2 \mid jm) = j(j+1) \mid jm) \ . \qquad (15)$$

Equations (2, 14, 15) *represent* the standard 'abstract' angular momentum states $\mid jm)$, and show how to manipulate them via differentiation. In particular, for $j = 1/2$ and $\gamma = \alpha_1 \beta_2 - \alpha_2 \beta_1$, $\mathbf{j}^2 \gamma$ is $[(j_{1x} + j_{2x})^2 + (j_{1y} + j_{2y})^2 + (j_{1z} + j_{2z})^2][\alpha_1 \beta_2 - \alpha_2 \beta_1] = 0$; so $\gamma \sim \mid 00)$ and γ is an invariant, see §1. Again for $j = 1/2$, $j_- \alpha = \beta$ etc., so $\alpha \sim \mid 1/2, 1/2)$ etc.; similarly $(j_{1-} + j_{2-})(\alpha_1 + \alpha_2) = (\beta_1 \partial / \partial \alpha_1 + \beta_2 \partial / \partial \alpha_2)(\alpha_1 + \alpha_2) = (\beta_1 + \beta_2)$, so $(\alpha_1 + \alpha_2) \sim \mid 1/2, 1/2)$, and thus $\alpha_1 + \alpha_2 \sim \alpha$ etc., as required in §1.

We have implied that (3) is 'reasonable'. Let $n = 1 ... 3$, $j = j_3$, $x = (\mathbf{j}, \alpha, \beta)$, $x_3 = x_1 + x_2$, $n^* = (j_n + m_n)!(j_n - m_n)!$ and let f_3 and f_{12} mean f evaluated in states $\mid j_3 m_3)$ and states $\mid j_1 m_1) \mid j_2 m_2)$. Then our logic flow is: $\mid jm) \equiv \mid jm)$,

$\mid 00) \mid jm) \equiv \mid 00) \mid jm)$, $[\mid 00) \mid jm)]_3 \equiv [\mid 00) \mid jm)]_{12}$,

$\mid 00)_3 \mid jm)_3 = N \mid 00)_{12} \mid jm)_{12}$, $\quad \mid 00)_3 = 1$,

$\mid 00)_{12} = (\alpha_1 \beta_2 - \alpha_2 \beta_1)^q$,

$\mid jm)_3 = \mid j_3 m_3) = \alpha_3^{j_3 + m_3} \beta_3^{j_3 - m_3} / \sqrt{3^*}$, $\mid jm)_{12} = \mid j_3 m_3)_{12}$

$= (\alpha_1 + \alpha_2)^{j_3 + m_3} (\beta_1 + \beta_2)^{j_3 - m_3} / \sqrt{3^*}$

$\alpha_3^{j_3+m_3} \beta_3^{j_3-m_3} = N(\alpha_1 \beta_2 - \alpha_2 \beta_1)^q (\alpha_1 + \alpha_2)^{j_3+m_3} (\beta_1 + \beta_2)^{j_3-m_3}$.

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