Triple-pendulum Model for Studying the Vibration of Multi-Degree-of-freedom Systems

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Abstract

Assorted real life examples are indispensable to teaching, however, most engineering vibration textbooks are deficient in such examples, and most of them are exclusively mass-spring systems, especially the multi-degree-of-freedom case. To enrich lecture content, some triple-pendulum instances are presented. The parameters of these instances are tuned to have closed-form solutions. The principal vibration and free vibration of one instance are presented. These examples can be illustrated clearly and promptly in limited time and writing space.

Keywords: Pendulum, Vibration, Multi-degree-of-freedom.

Resumen

Una variedad de ejemplos de la vida real son indispensables para la enseñanza, sin embargo, la mayoría de los libros de texto de ingeniería de vibraciones son deficientes en tales ejemplos, y la mayoría de ellos son exclusivamente los sistemas de masa-resorte, sobre todo el multi-grado-de-casos de libertad. Para enriquecer el contenido de la clase, se presentan algunas instancias del péndulo triple. Los parámetros de estas instancias están ajustados para tener soluciones de forma-cerrada. Se presentan la vibración principal y la vibración libre de un ejemplo. Estos ejemplos se pueden ilustrar clara y prontamente en tiempo limitado y en el espacio escrito.

Palabras clave: Péndulo, Vibración, Multi-grado-de-libertad.

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I. INTRODUCTION

In lectures on vibration, versatile examples and exercises are indispensable. However, the available material in most engineering textbooks [1, 2, 3, 4] is predominantly based on the mass-spring system, especially the case of the multi-degree-of-freedom systems. We have presented examples based the discrete mass and massless beams [5].

The pendulum has been an essential component in physics at least from the times of Galileo's finding the isochrony of a pendulum [6]. Even today, research on pendulums is still very active, for example, the chaos and stability of pendulums [7, 8]. Recently, we investigated the stability of the human's stance posture using the model of inverted pendulum with a coiled spring [9], which has drawn attention from some researchers[10, 11]. Gauld has written a systemic review on pendulums in the physics education [12].

The simple pendulum is a canonical example of a single degree-of-freedom (DOF) system in engineering vibration textbooks, while the double pendulum is frequently included in textbooks as an example of two-DOF systems. Even so, in such textbooks, most examples with DOF greater than two are limited to mass-spring systems, because of the complexity of alternative examples. A classroom example should be concise in both time and space, so that it can be finished in the class time and the limited space of a blackboard or a few multimedia slides. Further, too complicated or lengthy examples become difficult to grasp the main theme and record in note form.

In this paper, some triple-pendulum instances will be presented. The triple-pendulum is a nonlinear system. It can be linearized for small amplitude vibration, which is highlighted in an engineering vibration course. Even after linearization, to solve a general problem necessitates an iterative approach, which is not suit for a classroom lecture.

The triple-pendulum instances given here have closed-form solutions, because of their elaborate parameters. They can be lectured in a timely and efficiently manner in classroom lecture.

II. PHYSICAL MODEL

The triple-pendulum is shown in Fig. 1. For enhancing diversity, a spring is attached to m_3. The three massless trusses have lengths of l_1, l_2 and l_3 individually. The three discrete mass are m_1, m_2 and m_3 individually. We assume that the spring does have deformation when all the three trusses are vertical.

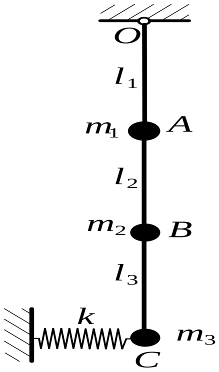

FIGURE 1. The model of triple-pendulum with a spring.

III. MATHEMATICAL MODEL

The Lagrange's equations are used here, so the generalized displacements, θ_1, θ_2 and θ_3 are selected as in Fig. 2.

The velocity analysis is illustrated in Fig. 2, thus the kinetics for three masses are:

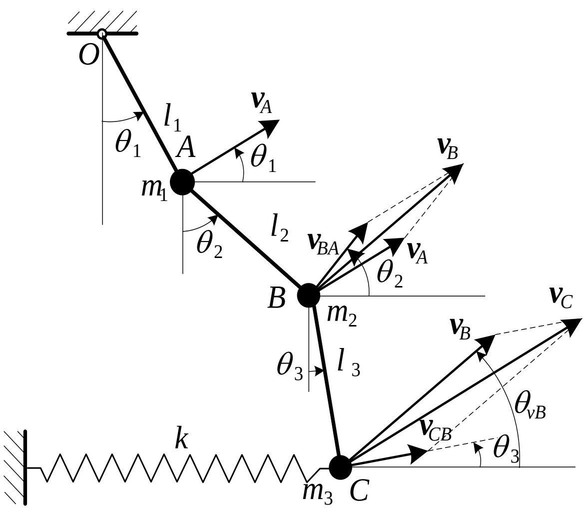

FIGURE 2. Movement analysis.

$$T_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 (l_1 \dot{\theta}_1)^2,$$

$$T_2 = \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_2 [(l_1 \dot{\theta}_1)^2 + (l_2 \dot{\theta}_2)^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1)]$$

$$\approx \frac{1}{2} m_2 [(l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2)^2],$$

$$T_3 = \frac{1}{2} m_2 v_3^2 = \frac{1}{2} m_3 [v_B^2 + (l_3 \dot{\theta}_3)^2 + 2 v_B l_3 \dot{\theta}_3 \cos(\theta_{vB} - \theta_3)]$$

$$\approx \frac{1}{2} m_3 [(l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + l_3 \dot{\theta}_3)^2],$$

where we have used the approximation: $1 - \cos\theta_1 \approx \frac{\theta_1^2}{2}$, $1 - \cos\theta_2 \approx \frac{\theta_2^2}{2}$, $1 - \cos\theta_3 \approx \frac{\theta_3^2}{2}$, $\cos(\theta_2 - \theta_1) \approx 1$ and $\cos(\theta_3 - \theta_{vB}) \approx 1$ for the small vibration amplitude.

The total kinetic energy is

$$T = T_1 + T_2 + T_3$$
$$= \frac{1}{2} m_1 (l_1 \dot{\theta}_1)^2 + \frac{1}{2} m_2 [(l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2)^2] + \frac{1}{2} m_3 [(l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + l_3 \dot{\theta}_3)^2].$$

The potential energy is

$$U = \frac{1}{2} k (l_1\theta_1 + l_2\theta_2 + l_3\theta_3)^2 +$$
$$m_1 l_1 \mathrm{g}(1 - \cos\theta_1) + m_2 \mathrm{g}[l_1(1 - \cos\theta_1) + l_2(1 - \cos\theta_2)] +$$
$$m_3 \mathrm{g}[l_1(1 - \cos\theta_1) + l_2(1 - \cos\theta_2) + l_3(1 - \cos\theta_3)]$$
$$= \frac{1}{2} k (l_1\theta_1 + l_2\theta_2 + l_3\theta_3)^2 + \frac{m_1 l_1 \mathrm{g}}{2} \theta_1^2 +$$
$$\frac{m_2 l_2 \mathrm{g}}{2} (\theta_1^2 + \theta_2^2) + \frac{m_3 l_3 \mathrm{g}}{2} (\theta_1^2 + \theta_2^2 + \theta_3^2).$$

The Lagrange's equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}_j}\right) - \frac{\partial T}{\partial \theta_j} + \frac{\partial U}{\partial \theta_j} = 0 \qquad (j = 1, 2, 3). \qquad (1)$$

All the quantities involved are as follows,

$$\frac{\partial T}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2) + m_3 l_1 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + l_3 \dot{\theta}_{31})$$
$$= (m_1 + m_2 + m_3) l_1^2 \dot{\theta}_1 + (m_2 + m_3) l_1 l_2 \dot{\theta}_2 + m_3 l_1 l_3 \dot{\theta}_{31},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = (m_1 + m_2 + m_3) l_1^2 \ddot{\theta}_1 + (m_2 + m_3) l_1 l_2 \ddot{\theta}_2 + m_3 l_1 l_3 \ddot{\theta}_3,$$

$$\frac{\partial T}{\partial \dot{\theta}_2} = m_2 l_2 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2) + m_3 l_2 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + l_3 \dot{\theta}_3)$$
$$= (m_2 + m_3) l_2 l_1 \dot{\theta}_1 + (m_2 + m_3) l_2^2 \dot{\theta}_2 + m_3 l_2 l_3 \dot{\theta}_3,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = (m_2 + m_3) l_2 l_1 \ddot{\theta}_1 + (m_2 + m_3) l_2^2 \ddot{\theta}_2 + m_3 l_2 l_3 \ddot{\theta}_3,$$

$$\frac{\partial T}{\partial \dot{\theta}_3} = m_3 l_3 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + l_3 \dot{\theta}_3),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}_3}\right) = m_3 l_1 l_3 \ddot{\theta}_1 + m_3 l_2 l_3 \ddot{\theta}_2 + m_3 l_3^2 \ddot{\theta}_3,$$

$$\frac{\partial U}{\partial \theta_1} = k l_1 (l_1\theta_1 + l_2\theta_2 + l_3\theta_3) + (m_1 l_1 + m_2 l_2 + m_3 l_3) g \theta_1$$
$$= [k l_1^2 + (m_1 l_1 + m_2 l_2 + m_3 l_3) g] \theta_1 + k l_1 l_2 \theta_2 + k l_1 l_3 \theta_3,$$

$$\frac{\partial U}{\partial \theta_2} = kl_2(l_1\theta_1 + l_2\theta_2 + l_3\theta_3) + m_2 l_2 g\theta_2 + m_3 l_3 g\theta_2$$
$$= kl_1 l_2\theta_1 + [kl_2^2 + (m_2 l_2 + m_3 l_3)g]\theta_2 + kl_2 l_3\theta_3 ,$$

$$\frac{\partial U}{\partial \theta_3} = kl_3(l_1\theta_1 + l_2\theta_2 + l_3\theta_3) + m_3 l_3 g\theta_3$$
$$= kl_1 l_3\theta_1 + kl_2 l_3\theta_2 + (kl_3^2 + m_3 l_3 g)\theta_3 .$$

Substituting all the above quantities into Eq. (1) leads to

$$[M]\{\ddot{\theta}\} + [K]\{\theta\} = \{0\} . \tag{2}$$

Where

$$\{\theta\} = \begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix}, [M] = \begin{bmatrix}(m_1+m_2+m_3)l_1^2 & (m_2+m_3)l_1 l_2 & m_3 l_1 l_3\\ (m_2+m_3)l_1 l_2 & (m_2+m_3)l_2^2 & m_3 l_2 l_3\\ m_3 l_1 l_3 & m_3 l_2 l_3 & m_3 l_3^2\end{bmatrix},$$

$$[K] = \begin{bmatrix}kl_1^2 + (m_1 l_1 + m_2 l_2 + m_3 l_3)g & kl_1 l_2 & kl_1 l_3\\ kl_1 l_2 & kl_2^2 + (m_2 l_2 + m_3 l_3)g & kl_2 l_3\\ kl_1 l_3 & kl_2 l_3 & kl_3^2 + m_3 l_3 g\end{bmatrix}.$$

IV. INSTANCES

The eigenvalue problem of Eq. (2) necessitates an iterative computation, which is not tractable in a theoretical course. Here we set $m_1 = m_2 = m_3 = m,\ l_1 = l_2 = l_3 = l$, and give the follow instances with closed-form solutions.

Instance 1: $k = 0$

The mass and stiff matrixes are

$$[M] = ml^2\begin{bmatrix}3 & 2 & 1\\ 2 & 2 & 1\\ 1 & 1 & 1\end{bmatrix},\ [K] = mgl\begin{bmatrix}3 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

The eigenvalues equation is

$$|\text{-}\lambda[M] + [K]| = m\left\|\begin{bmatrix}3g-3l\lambda & -2l\lambda & -l\lambda\\ -2l\lambda & 2g-2l\lambda & -l\lambda\\ -l\lambda & -l\lambda & g-l\lambda\end{bmatrix}\right\| = 0 .$$

That is

$$-l^3\lambda^3 + 9gl^2\lambda^2 - 18g^2 l\lambda + 6g^3 = 0 .$$

We can solve out

$$\lambda_1 = \left(3\text{-}\sqrt{3}\cos\alpha - 3\sin\alpha\right)\frac{g}{l} \approx 0.4158\frac{g}{l},$$
$$\lambda_2 = \left(3\text{-}\sqrt{3}\cos\alpha + 3\sin\alpha\right)\frac{g}{l} \approx 2.2943\frac{g}{l},$$
$$\lambda_3 = \left(3\text{+}2\sqrt{3}\cos\alpha\right)\frac{g}{l} \approx 6.2899\frac{g}{l},$$

where $\alpha = \dfrac{\arctan\sqrt{2}}{3} \approx 0.3184$. The expressions for the eigenvectors are rather lengthy, thus only the approximate digital values are list as follows:

$$\phi_1 = \{0.4331, 0.5597, 0.7065\}^{\mathrm{T}} ,$$
$$\phi_2 = \{0.3814, 0.1346, -0.9146\}^{\mathrm{T}} ,$$
$$\phi_3 = \{0.4282, -0.7938, -0.3701\}^{\mathrm{T}} .$$

Instance 2: $k = \dfrac{mg}{l}$

The stiff matrix is

$$[K] = mgl\begin{bmatrix}4 & 1 & 1\\ 1 & 3 & 1\\ 1 & 1 & 2\end{bmatrix}.$$

The eigenvalues are

$$\lambda_1 = \frac{g}{l},$$
$$\lambda_2 = \frac{9\text{-}\sqrt{13}}{2}\frac{g}{l},$$
$$\lambda_3 = \frac{9\text{+}\sqrt{13}}{2}\frac{g}{l},$$

The corresponding eigenvectors are

$$\phi_1 = \{1,\quad 1,\quad 0\}^{\mathrm{T}} ,$$
$$\phi_2 = \{1,\quad \frac{\sqrt{13}-3}{4}, -\frac{\sqrt{13}+2}{2}\}^{\mathrm{T}} ,$$
$$\phi_3 = \{1, -\frac{\sqrt{13}+3}{4},\quad \frac{\sqrt{13}-2}{2}\}^{\mathrm{T}} .$$

Instance 2: $k = \dfrac{4mg}{l}$

The mass and stiff matrixes are

$$[M] = ml^2\begin{bmatrix}3 & 2 & 1\\ 2 & 2 & 1\\ 1 & 1 & 1\end{bmatrix},\ \ [K] = mgl\begin{bmatrix}7 & 4 & 4\\ 4 & 6 & 4\\ 4 & 4 & 5\end{bmatrix}.$$

The eigenvalues are

$$\lambda_1 = (4-\sqrt{6})\frac{g}{l},$$
$$\lambda_2 = 5\frac{g}{l},$$
$$\lambda_3 = (4+\sqrt{6})\frac{g}{l}.$$

The corresponding eigenvectors are

$$\phi_1 = \{1, \frac{\sqrt{6}-1}{2}, -\frac{\sqrt{6}}{2}\}^{\mathrm{T}},$$
$$\phi_2 = \{1, -1, -2\}^{\mathrm{T}},$$
$$\phi_3 = \{1, -\frac{\sqrt{6}+1}{2}, \frac{\sqrt{6}}{2}\}^{\mathrm{T}}.$$

It is very interesting to note that the first order eigenvector has a node. Normally, for a mass-spring system, the first order eigenvector (modal vector) does not have a node.

Instance 4: $k = 6\frac{mg}{l}$

The stiff matrix is

$$[K] = mgl\begin{bmatrix} 9 & 6 & 6 \\ 6 & 8 & 6 \\ 6 & 6 & 7 \end{bmatrix}.$$

The eigenvalues are

$$\lambda_1 = \frac{9\text{-}\sqrt{33}}{2}\frac{g}{l},$$
$$\lambda_2 = 6\frac{g}{l},$$
$$\lambda_3 = \frac{9\text{+}\sqrt{33}}{2}\frac{g}{l}.$$

The corresponding eigenvectors are

$$\phi_1 = \{1, \frac{\sqrt{33}-3}{4}, -\frac{\sqrt{33}-3}{2}\}^{\mathrm{T}},$$
$$\phi_2 = \{1, -\frac{3}{2}, 0\}^{\mathrm{T}},$$
$$\phi_3 = \{1, -\frac{\sqrt{33}+3}{4}, \frac{\sqrt{33}+3}{2}\}^{\mathrm{T}}.$$

IV. PRINCIPAL VIBRATION

Discussions in this and the next sections are pertaining to instance 2 from section 3. Denote the modal shape matrix (eigenvector matrix) as

$$[\Phi] = [\phi_1, \phi_2, \phi_3] = \begin{bmatrix} 1 & 1 & 1 \\ \frac{\sqrt{6}-1}{2} & -1 & -\frac{\sqrt{6}+1}{2} \\ -\frac{\sqrt{6}}{2} & -2 & \frac{\sqrt{6}}{2} \end{bmatrix}.$$

It can be verified that the matrix $[\Phi]$ indeed diagonalize the mass and stiff matrices as

$$[\Phi]^{\mathrm{T}}[M][\Phi] = ml^2\begin{bmatrix} 3+\frac{\sqrt{6}}{2} & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3-\frac{\sqrt{6}}{2} \end{bmatrix},$$

$$[\Phi]^{\mathrm{T}}[K][\Phi] = mgl\begin{bmatrix} 9-\sqrt{6} & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 9+\sqrt{6} \end{bmatrix}.$$

Introducing the modal transform $\{\theta\} = [\Phi]\{q\}$ arrives at the uncoupling equations as

$$\left.\begin{aligned} \frac{6+\sqrt{6}}{2}ml^2\ddot{q}_1 + (9-\sqrt{6})mgl\,q_1 = 0 \\ 5ml^2\ddot{q}_2 + 25mgl\,q_2 = 0 \\ \frac{6-\sqrt{6}}{2}\ddot{q}_3 + (9+\sqrt{6})mgl\,q_3 = 0 \end{aligned}\right\}, \quad (3)$$

where q_1, q_2, and q_3, are modal variables.

The solutions of Eq. (3) are the following sinusoidal functions

$$q_i(t) = q_i(0)\sin p_i t + \frac{\dot{q}_i(0)}{p_i}\sin p_i t \qquad (i = 1 \sim 3), \quad (4)$$

where $p_i = \sqrt{\lambda_i}(i = 1 \sim 3)$ are the modal frequencies. The solutions are depicted in the first row of Figure 3.

The modal shapes are shown in the third row of Figure 3. It should be noted that they are not drawn as per the vectors ϕ_1, ϕ_2 and ϕ_3, because the angular values do not have visual meanings. The absolute geometric positions in the modal shape plots are calculated according to the angular values. This also explains why there is a node in ϕ_1, because ϕ_1 is only a mathematical eigenvector. The mathematical eigenvector had better be transformed to the position in the physics space, so that a vivid image can be discerned easily.

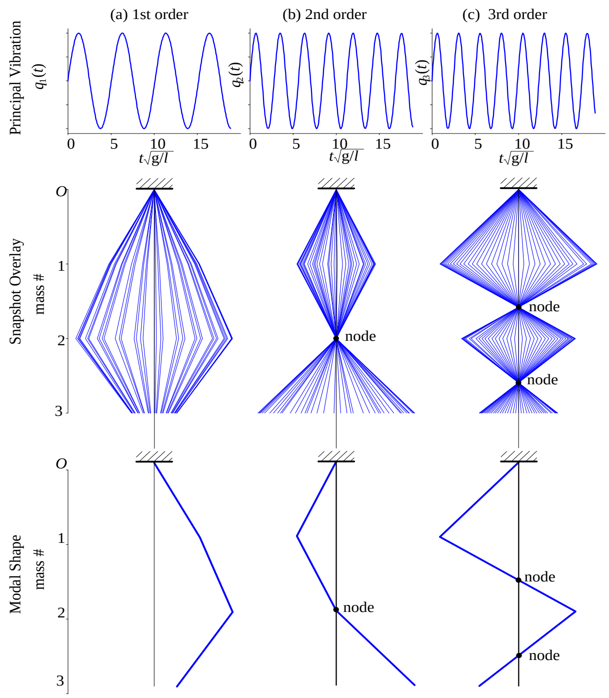

FIGURE 3. Principal vibrations.

The second row in Fig. 3 shows the overlap of system profile snapshot taking at many instants. The stable nodes and vibration patterns (modes) can be discerned easily. The number of nodes increases with the modal orders.

V. FREE VIBRATION

For arbitrary initial conditions $\{\theta(0)\}$ and $\{\dot\theta(0)\}$, those for Eq. (4) are

$$\{q(0)\}=[\Phi]^{-1}\{\theta(0)\},\{\dot q(0)\}=[\Phi]^{-1}\{\dot\theta(0)\}.$$

For example

$$\{\theta(0)\}=\begin{Bmatrix}15\\-10\\-10\end{Bmatrix},\qquad \{\dot\theta(0)\}=\begin{Bmatrix}13\\-8\\-6\end{Bmatrix},$$

we can solve out the

$$\{q(0)\}=\begin{Bmatrix}5\\5\\5\end{Bmatrix},\{\dot q(0)\}=\sqrt{\frac{g}{l}}\begin{Bmatrix}5\\3\\5\end{Bmatrix}.$$

Thus the free response for each modal order as per Eq. (4) is

$$q_1(t)=5\cos p_1t+\frac{5}{p_1}\sqrt{\frac{g}{l}}\sin p_1t,$$

$$q_2(t)=5\cos p_2t+\frac{3}{p_2}\sqrt{\frac{g}{l}}\sin p_2t,$$

$$q_3(t)=5\cos p_3t+\frac{5}{p_3}\sqrt{\frac{g}{l}}\sin p_3t.$$

According to the transform $\{\theta\}=[\Phi]\{q\}$, the free response in the physical space is

$$\{\theta(t)\}=\frac{5}{2}\left(\cos p_1t+\frac{\sqrt{54+6\sqrt{33}}}{12}\sin p_1t\right)\begin{Bmatrix}2\\\sqrt{6}-1\\-\sqrt{6}\end{Bmatrix}+$$

$$\left(5\cos p_2t+\frac{\sqrt{6}}{2}\sin p_2t\right)\begin{Bmatrix}1\\-1\\-2\end{Bmatrix}+$$

$$\frac{5}{2}\left(\cos p_3t+\frac{\sqrt{54-6\sqrt{33}}}{12}\sin p_3t\right)\begin{Bmatrix}2\\-\sqrt{6}-1\\\sqrt{6}\end{Bmatrix}. \tag{5}$$

Their trajectories are depicted in Fig. 4 (a), (b), (c) individually. These curves are not sinusoidal any longer, and look rather complicated. However Eq. (5) indicates that these complicated curves are composed by the simple principal vibrations.

Fig. 4d is parallel to Fig. 3d, the mass positions in the physical space. This plot shows that the free vibration does not have a stable node and pattern like the principal vibrations

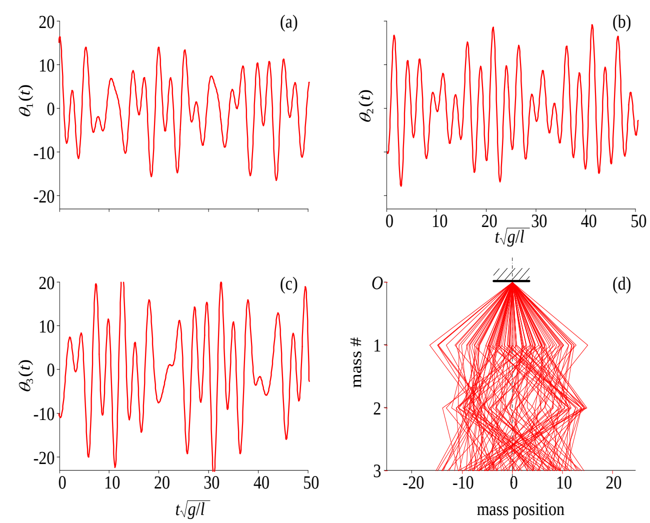

FIGURE 4. Free vibrations.

VI. CONCLUSION

The triple-pendulum with a spring was investigated using Lagrange's equations and linearization. Some pendulum's instances with closed-form value solutions are presented. These instances can be used as lecture examples or exercises.

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