# Teaching Nonlinear Dynamics and Chaos for Beginners

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**Abstract**

We describe a course in Nonlinear Dynamics for undergraduate students of the first years of Chemical Engineering, Environmental Sciences and Computer Sciences. An extensive use of computational tools, the internet and laboratory experiments are key ingredients of the course. Even though their previous background in physics and mathematics might be limited, our experience shows that an appropriate selection of the contents with the use of some conceptual introductory ideas and multimedia techniques are an excellent way to introduce Nonlinear Dynamics and Chaos for beginners. The active participation of the students and the extraordinary interest arisen in them has been surprising.

**Keywords:** Physics Education, Nonlinear Dynamics and Chaos.

**Resumen**

Describimos un curso de Dinámica No Lineal para estudiantes de los primeros cursos de las titulaciones de Ingeniería Química, Ciencias Ambientales e Informática. El uso extensivo de herramientas computacionales, internet y prácticas de laboratorio son los ingredientes clave de este curso. Aún siendo sus conocimientos previos en física y matemáticas limitados, nuestra experiencia muestra que una selección adecuada de los contenidos junto con algunos conceptos introductorios y técnicas multimedia son una forma excelente para introducir la Dinámica No Lineal y Teoría del Caos para principiantes. La activa participación de los estudiantes y el extraordinario interés alcanzado en ellos han sido sorprendentes.

**Palabras clave:** Enseñanza de la Física, Dinámica No Lineal y Caos.

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## I. INTRODUCTION

Nonlinear Dynamics and Chaos has been developed in the past years as a new emergent field in Physics with an interdisciplinary character. Introductory courses on this field are quite usual for graduate courses in sciences, but finding them as part of the education for undergraduate students in sciences and engineering is far more difficult, with the exception of Mathematics and Physics degrees. Our aim in this paper is to describe a course on Nonlinear Dynamics for undergraduate students with very different backgrounds that has been offered as an elective subject with growing success during the last 10 years in all science and engineering degrees at our university. What makes our course singular is that the students that have attended it have very different profiles, but most of them are students in Chemical Engineering, Environmental Sciences and Computer Sciences. Our experience has shown us that Nonlinear Dynamics is found as a very interesting subject by this heterogeneous collection of students, due to the global vision of the dynamical phenomena offered. On the other hand, we have learned that it is possible to make an introductory course on an specific field of physics, such as Nonlinear Dynamics, in such a way that it can be found interesting for students outside the degrees of Mathematics and Physics, which do not necessarily have a strong background on them either.

The main goal of this course is to introduce and describe the chaotic phenomena in physical systems by only using a minimum background in physics and mathematics. We try to show a general overview of nonlinear dynamical systems and their applications in science and technology. Numerical simulations have been a basic point in the development of Nonlinear Dynamics, and they continue to be a very important tool for beginners in this field, as long as they allow to understand dynamical phenomena without having a deep mathematical knowledge of the involved mechanisms. Thus, throughout this course the use of JAVA applets simulations and other software tools such as DYNAMICS [1] and CHAOS FOR JAVA [2] play a key role. Another interesting and important part of this course is the nonlinear physics laboratory, where the students are able to visualize nonlinear and chaotic phenomena in real experiments in the laboratory. During the last years we have made use of some of the ideas explained in Refs. [3, 4], where different laboratory experiments in Nonlinear Physics are shown. All this allows us to introduce the main concepts of Nonlinear Dynamics in a visual way without needing a detailed exposition of the mathematical aspects of the theory.

The structure of this paper is organized as follows. In Sec. II we introduce the main contents of this course. Section III shows the goals of this course and the methodology carried out in it. Conclusions are presented in Sec. IV.

## II. CONTENTS

To decide the contents of an introductory course on a wide field of physics such as Nonlinear Dynamics is not an easy task. An important first decision that needs to be made before planning the structure of the course comes from the fact that in Nonlinear Dynamics, both continuous time and discrete time dynamical systems play a key role. There might be reasons for deciding to introduce first one or another. However, our experience tells us that introducing first the discrete time dynamical systems is a good choice. Our students usually do not have a background on differential equations, and with discrete time dynamical systems the concepts of temporal evolution and orbits are easy to understand. On the other hand, the first basic concepts on Nonlinear Dynamics, like the concept of *chaos*, can be easily introduced by using simple paradigmatic discrete dynamical systems such as the logistic map.

The selection of the contents should always be a result of the previous decision on the goals. Two fundamental aspects are needed to be considered to design a teaching plan: the methodology and the organization of the contents. In the planning and the design of the course we cannot forget either the duration of the course nor the background and previous knowledge of the students to whom the course is addressed. Considering the main goal of our course, which is to give an introductory course of Nonlinear Dynamics with stress to applications to different fields, we have divided our course in just 10 chapters that deal with a big part of Nonlinear Dynamics, which is shown now. After that, we make a brief description of each of its parts:

1. Introduction to Nonlinear Dynamics and Chaos
2. Discrete Dynamical Systems: 1D Maps
3. Two Dimensional Maps
4. Concepts in Dynamical Systems Theory
5. Elementary Bifurcation Theory
6. Chaotic Dynamical Systems
7. Lyapunov Exponents
8. Fractals and Fractal Dimension
9. Hamiltonian Chaos
10. Introduction to Nonlinear Time Series Analysis

**1. Introduction to Nonlinear Dynamics and Chaos:** In this chapter we make an introduction and course description, as long as a historical overview of the subject. The first elementary notions of the concept of a dynamical system is given with the help of a simple physical system: the pendulum. Fractals are also presented here. During this first chapter we underline that this is an emergent and interdisciplinary field of physics, and it allows to obtain a *dynamical view* of the world. *Bibliography*: Chapter 1 of [5].

**2. Discrete Dynamical Systems. One-dimensional maps:** Here we introduce some of the elementary notions of Nonlinear Dynamics, such as the notion of dynamical system, bifurcation and chaotic behavior, by making use of simple discrete dynamical systems. First, a linear discrete system whose dynamics can be easily understood is given. After this, via the logistic map, it is shown that the presence of nonlinearities can make the dynamics more complicated. We stress the influence of parameters on the global dynamics with the help of this map. Moreover, we explain some geometrical methods to obtain useful information about the system, such as cobweb maps [5]. *Bibliography:* Chapter 5 of [6], chapter 10 of [5] and chapter 1 of [7].

**3. Two-dimensional maps:** Once the students are familiar with one-dimensional maps, it is the moment to introduce two-dimensional discrete dynamical systems. This allows to introduce notions that cannot be explained with one-dimensional maps, for example the classification of fixed points as *centers*, *sinks*, *sources* and *saddles.* This is first done by introducing simple two-dimensional linear maps, after which this notion is easily extended to nonlinear maps and illustrated with simulations of DYNAMICS [1]. With the concept of stable and unstable manifolds we proceed analogously: first we introduce the concept with the help of linear maps, and by using the DYNAMICS software we show how they look like for some paradigmatic nonlinear system, such as the Hénon map, both in simple situations and complicated ones, with homoclinic intersections. After this, in order to make more clear the connection between the two-dimensional maps and physical systems, we explain the bouncing ball model, whose dynamics is described by a two-dimensional map and that presents a wide variety of behaviors. *Bibliography:* Chapter 2 of [8], chapter 5 of [6] and chapter 1 of [7].

**4. Concepts in Dynamical Systems Theory:** With the background earned by analyzing different dynamical phenomena and different concepts with maps, we can now introduce some simple examples of continuous-time dynamical systems. First, one-dimensional continuous time systems, such as the logistic equation, are introduced. The simple dynamics of this kind of systems is analyzed in certain detail, emphasizing the geometrical point of view (that allows to understand the system's dynamics without solving the differential equation). After this, we give an example of a higher dimensional continuous-time dynamical system: a mass spring system, which allows us to give a definition of phase space for this system. The

Lotka-Volterra model is introduced as a nonphysical model that is a dynamical system with applications in different areas, for example, in Ecology, Economy, dynamics of web sites in internet, etc. Some basic notions on how to solve differential equations numerically are also given. *Bibliography:* Chapters 2 and 4 of [5] and chapters 4 and 5 of [7].

**5. Elementary bifurcation theory:** Our objective here is to give a clear concept of bifurcation and give some examples of this phenomenon. The notion of bifurcation has already been introduced when a description of one-dimensional maps and of Feigenbaum bifurcation diagram was done in the first part of the course. Thus, by now the students have an intuitive notion of how a variation of a parameter can change in a qualitative way the dynamics of the system. In this part, we do a more quantitative approach to this phenomenon making use of the dynamical systems that can be analyzed more easily: one-dimensional flows. The geometrical tools developed in the last chapter for these systems allows classifying some of the most important bifurcations, which are linked with some examples from physics and using numerical simulations (see Ref. [9]) that allow to visualize in a very graphic way the different types of bifurcations. Through simulations students can appreciate how the phase space is transformed as one of the parameters is varied, and some typical phenomena such as the appearance and destruction of fixed points or the period-doubling bifurcation (see Fig. 1) can be easily visualized and fully understood. *Bibliography*: Chapter 3 of [5] and chapter 3 of [6].

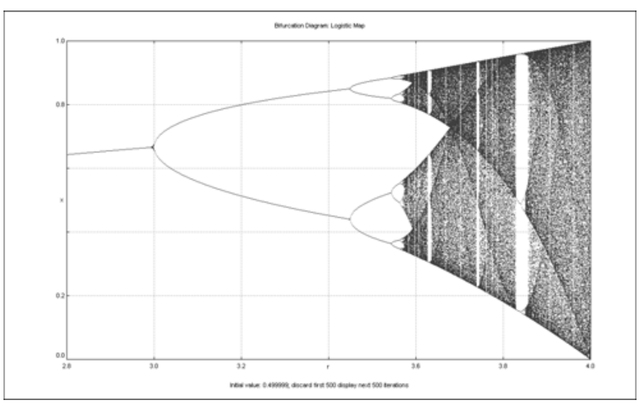


**FIGURE 1.** Figure showing a typical period-doubling bifurcation in the logistic map (Figure obtained from [2]).

**6. Chaotic Dynamical Systems:** In this chapter we give some basic notions of chaotic behavior for maps and flows. We introduce some simple maps and flows that display chaotic behavior. A special attention is paid on the Hénon map as a paradigmatic example of a two-dimensional chaotic map possessing a chaotic attractor. Examples of chaotic flows are also introduced. For two-dimensional flows some nonlinear driven chaotic oscillators are analyzed, and finally some three-dimensional flows such as the Lorenz model are considered. The analysis of the chaotic dynamics of all these systems are performed through numerical simulations with the computer, and different techniques of visualization of their dynamical behavior are used such as the study of the return maps and the basins of attraction, the transformations on the attractors that take place when the parameters of the system are varied, the evolution in time of the dynamical variables, the study of the Poincaré map, the dynamics on the phase space, etc. *Bibliography*: Chapters 5 and 9 of [8] and chapters 9 and 12 of [5].

**7. Lyapunov Exponents:** Once the students have a qualitative notion of chaos, we can now give a more quantitative notion of chaos. The notion of sensitive dependence on the initial conditions has been stated as one of the fingerprints of chaotic motion through some simple numerical examples in the previous chapter. One of the simplest quantitative methods to know if a dynamical system is chaotic or not is the calculation of Lyapunov exponents. In this chapter it is explained how this quantity is closely related to the sensitive dependence on the initial conditions of chaotic systems and how can it be computed. Examples of calculation of the Lyapunov exponents are shown through the help of numerical simulations. It is especially important to show diagrams where the largest Lyapunov Exponent is shown against some parameter of the system, as it can be useful to illustrate in a very simple way the transition between periodic and chaotic motion that may take place in a dynamical system as a parameter is varied. Some beautiful applets illustrating these phenomena can be found in Ref. [2]. *Bibliography:* Chapters 5 and 9 of [8] and chapter 9 of [6].

**8. Fractals and Fractal Dimension:** The main goal of this chapter is to introduce the notion of fractal set and its connection with dynamical systems. Note that some simple examples of fractals, such as the Cantor set, have already appeared in a natural way in previous chapters, for example when the Feigembaum's bifurcation diagram was exposed and the escape dynamics of the slope three tent map was discussed. Here, a systematic exposition of some simple fractal sets is done, such as the Cantor set (see Fig. 2), the Von Koch curve and the Sierpinski triangle, showing the algorithms used to build them. After this, the notion of fractal dimension is also introduced. Furthermore, fractal dimensions of some simple fractals are computed. A special attention is paid to the study of connections between fractals and dynamical systems, and some examples in physics where fractals structures arise are also given. Furthermore, the consequences of the appearance of such fractals structures on the predictability of the future state of a dynamical system are discussed, for example when fractal basin boundaries do appear. *Bibliography:* Chapter 9 of [6], chapter 4 of [8], chapter 11 of [5].

**FIGURE 2**. Figure of the algorithm to build the Cantor set.

**9. Hamiltonian Chaos:** In this chapter we provide the elements for understanding chaotic conservative systems. Through a digression about the concept of friction or energy dissipation in a physical system, the dynamical systems are classified as dissipative and conservative systems. The pendulum model is very easy to use in this context and it shows clearly that the systems that preserve the energy do not possess attractors (hallmark of dissipative systems). Examples of different conservative dynamical systems in physics are discussed and through them a new kind of chaotic motion is introduced: hamiltonian chaos. Some simple examples as the four Christmas balls model [10] are given, where also fractal structures can be visualized. Nonlinear periodically driven oscillators in absence of dissipation are also of great help when one wants to visualize Hamiltonian chaos. From the point of view of discrete dynamical systems, hamiltonian discrete systems are introduced, where area is preserved and Liouville's theorem applies, and the main concepts of the transition to hamiltonian chaos is illustrated by using the Chirikov's standard map, a paradigmatic system of this type. Our computational approach is similar to the approach described in Ref. [11]. *Bibliography:* Chapter 8 of [6] and chapter 8 of [12].

**10. Introduction to Nonlinear Time Series Analysis:** The contents of this course ends by introducing the elementary notions of nonlinear time series and some of their applications. Here, the basic notions of time series analysis are introduced, as well as the methods to detect stationarity and nonlinearity. The method to detect chaos in time series and to reconstruct the attractors via the embedding technique is briefly described. All this is illustrated by means of examples of different time series that appear in very different fields of science, from physics to medicine. Software packages for time series analysis might be used to explore the different aspects of time series analysis as shown in Ref. [13]. *Bibliography:* Chapter 6 of [7].

## III. OBJECTIVES, METHODOLOGY AND EDUCATIONAL ORGANIZATION

As we have already explained in the Introduction, the main objectives of this course are described in the following way. First, we introduce the basic notions on Nonlinear Dynamics and Chaos in order to provide to the students a suitable background to study the subject and to understand the different topics which are dealt during the course. These concepts are clarified by using several applications they have in science and technology. Furthermore, examples of the use in scientific computation and in applied sciences are shown. Finally some experiments are shown, in order to visualize different chaotic phenomena. All these elements give the students a good background and overview of the subject of Nonlinear Dynamics and Chaos.

The methodology is oriented in the use of different computational tools and software: Chaos for Java applets [2], Interactive Differential Equations [9] and software DYNAMICS [1], among others.

We complete the computational experiments with real experiments or demonstrations as the chaotic pendulum, double pendulum, Belousov-Zhabotinskii reaction, etc. A picture of the double pendulum laboratory experiment can be found in Fig. 3.

Tutorial lectures describing basic concepts are given during one semester as we explain as follows.

These tutorial lectures consist in sixty hours in a semester. Three hours per week for theoretical lessons in the classroom and one hour per week in the computer laboratory. It is also necessary to find some room for the experiments in the Nonlinear Physics Laboratory.

The prerequisites are minimal. Typically the knowledge of mathematics of a second year undergraduate student. Knowledge of a programming language is not compulsory, although it is an advantage. The main prerequisite is to be familiar with computers and with internet for regular users.

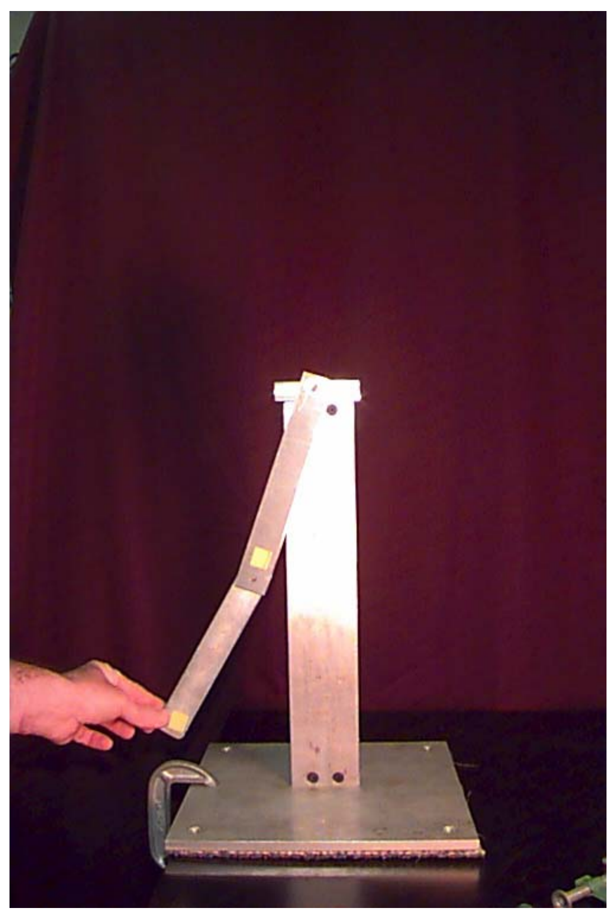

**FIGURE 3.** Picture showing the double pendulum laboratory experiment (Figure obtained from [14]).

## A. Methodology

The specific methodology applied for this course is based in the general criteria explained in this section and in the following summary:

- Exposition of the most important concepts through numerical simulations by computer. This takes place in the classroom or in the computer laboratory.
- Exposition in the classroom of theoretical foundations by using multimedia tools.
- Practical of virtual laboratory on Nonlinear Dynamics using simulations of JAVA applets (see Fig. 4).
- Practical in the Nonlinear Physics laboratory.
- Training with the software DYNAMICS, a specific program for dynamical systems, or other packages at disposal from Internet.
- Periodic homework for the students about the concepts explained in the classroom.
- The student has to write a supervised paper, which is explained in the following subsection.

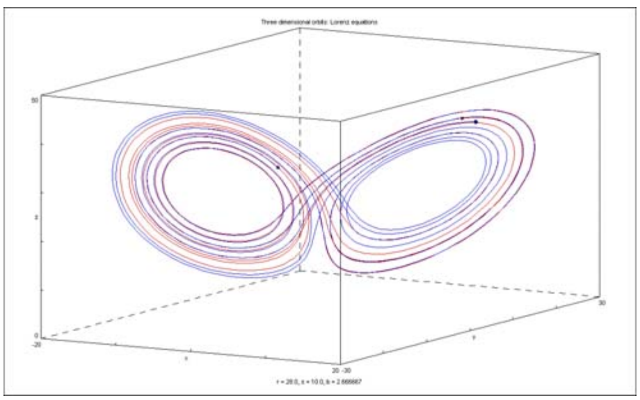

**FIGURE 4.** Visualization of the Lorenz attractor using Java applets (Figure obtained from [2]).

### A.1 Methods of Evaluation

The method of evaluation that we have used is basically founded in the continuous participation of the student in the process of learning. The complete procedure is as follows:

The students have to do some Homework that represents 15 % of the final grade. It mainly consists in answering a list of ten basic questions after a chapter is explained. They also have to prepare a supervised paper which represents 25%. The supervised paper is a very interesting idea with outstanding results which we comment in Subsect. III B 3. The final exam is 60% of the final grade. This is a comprehensive exam with ten questions, each one corresponding to a different chapter and where the basic ideas are asked. An optional work, Laboratory experiment, is offered to improve the final grade. This Laboratory Experiment has traditionally been to build the four Wada balls, that is, an experiment in which the topological Wada property is shown by means of a simple optical system [10]. A nice picture of this experiment is found in Fig. 5.

## B. Educational organization

To teach Nonlinear Dynamics is necessary to do simulations using computer and showing the students the properties of nonlinear and chaotic phenomena. On the other hand, some of the theoretical lessons are given in the computer lab, where the student explores new concepts already explained in the classroom, helping the students in the process of learning.

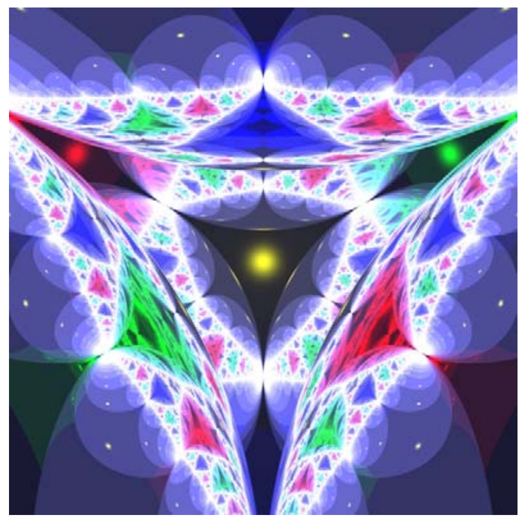

**FIGURE 5.**Picture of the simple optical system showing the Wada property (Figure obtained from [15]).

### B.1 Practical lessons

When teaching any topic in physics, in this specific case Nonlinear Dynamics, a very important core of this teaching are the practical lessons. The concept of practical lessons is broad and covers from the use of computer lab, a virtual Physics lab for Nonlinear Dynamics, Nonlinear Physics lab, to the resolution of practical exercises and demonstrations. A very interesting and useful idea for teaching Nonlinear Dynamics is to propose conceptual questions, simple periodical exercises, that students will solve by themselves to guarantee their knowledge and to stimulate their curiosity and creativity. Our experience has been great, and we have checked that the results have been surprising.

### B.2 Seminars

A useful tool from a methodological point of view are the seminars with the students. In these seminars they can expose the difficulties that they may have found when facing the theory by themselves, and it also allows the teacher to expose in a more detailed way those aspects of the subject that they may have not fully understood, allowing them to follow the future theoretical lessons more fluently.

These seminars are very useful to promote the capability of critical analysis and creation of the student, and they are also a perfect occasion to make the students comment, in a more relaxed way, about the difficulties in their learning of the subject.

### B. 3 Supervised paper

In this activity the student has the possibility of writing a supervised paper in Nonlinear Dynamics, about an issue that is related with the contents of the subject that can be chosen by the students according to their interest. The rules to be followed for the realization of the paper are not particularly strict, but there is one crucial rule that must be respected: the paper must be at most 3 pages long. Considering that there are a lot of information at their disposal, such as internet and all the books present at the library, the difficult task they have to face is to synthesize all the information in some fundamental ideas. Most part of the students show an amazing enthusiasm in performing this task. Moreover, the quality of many of these works was excellent, which makes us think that for some of the students this kind of activities offer them an opportunity to make a quality work that they usually cannot make in other disciplines.

The supervised paper could deal with very different subjects, according to the student's interest. If they choose a typical dynamical system for the supervised paper the most successful choices are Lorenz system, Hénon map, standard map, chaotic pendulum, Lotka-Volterra system, Hodgkin-Huxley system, the slope three tent map, logistic map, Belousov-Zhabotinskii reaction, Chua circuit and the Duffing oscillator. Among the different applications of this field the students prefer Chaos in Chemical Engineering, Chaos in Ecology, Cryptography and Chaos, controlling chaos, applications of Nonlinear Dynamics to Computer Sciences, Electronics and Communications Technologies and applications in Engineering and Technology. Moreover, the students are also interested in a varied theoretical and classical topics as The Newton method and fractals, Mandelbrot set, Smale horseshoe, Fractal basins, Lyapunov exponents, Hamiltonian chaos, algorithm to construct fractals, Cantor set, Chaos and randomness, among others.

## IV. CONCLUSIONS

It is noteworthy to comment on the results of the course. Perhaps one of the more outstanding and satisfactory results is to verify that at the end of the course almost all the students have become very familiar with many concepts in Nonlinear Dynamics and Chaos, taking into account that at the very beginning their knowledge was virtually none. Concerning the homework, it is also interesting to note that the response of the students has been excellent. Even though there is the risk of cheating, students have shown how to behave, and it is always a great stimulus to the progressive understanding of the contents of the course. It is remarkable the success of the supervised paper. It helps students to synthesize and to have a clear mind of one specific topic within the contents of the course.

We believe that the experience described in this paper is of an extraordinary interest. As a matter of fact, we are convinced that our experience can be transferred to many universities and the results obtained can be important in approaching the students to a new way of thinking and allowing them to see how through Nonlinear Dynamics many natural phenomena are interconnected. Students with no previous knowledge of Nonlinear Dynamics and with a rather elementary background in Physics and Mathematics are able to enjoy learning many interesting phenomena with the help of multimedia tools, computer simulations and traditional lectures. The degree of satisfaction of the students and teachers is worth trying it. Finally, we expect that the ideas outlined from our experience in this course might be useful as a way to introduce to a broad range of students of Universities from different countries to this subject, seldom mentioned in any major degree.

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