Three Important Taylor Series for Introductory Physics

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Abstract

Taylor expansions of the exponential exp(x), natural logarithm ln(1+x), and binomial series $(1+x)^n$ are derived to low order without using calculus. It is particularly simple to develop and graph the expansions to linear power in x. An example is presented of the application of the first-order binomial expansion to finding the electrostatic potential at large distances from an electric dipole. With a little extra work, the second-order expansions can be obtained starting from the familiar kinematics expression for the motion of a particle accelerating in one dimension, which instructively ties the mathematical development to physics concepts already presented in introductory courses.

Keywords: Exponential, logarithm, binomial series, electric dipole.

Resumen

Se derivan los desarrollos de Taylor de la exponencial exp (x), el logaritmo natural ln(1 + x), y la serie del binomio $(1 + x)^n$ para orden bajo sin utilizar cálculo. Especialmente las expansiones para potencia lineal en x son fáciles de desarrollar y graficar. Se presenta un ejemplo de la aplicación de la expansión de primer orden del binomio para encontrar el potencial electrostático a grandes distancias de un dipolo eléctrico. Con un poco de trabajo extra, se puede obtener la expansión de segundo orden a partir de la familiar expresión cinemática para el movimiento de una partícula acelerada en una dimensión, que vincula el desarrollo matemático con los conceptos de la física que se presentan en los cursos introductorios.

Palabras clave: Exponencial, logaritmo, serie binomial, dipolo eléctrico.

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I. INTRODUCTION

Approximating a binomial series by the sum of its first few terms is useful throughout an introductory physics course. Example applications [1, 2] include estimating square roots and derivatives, properties of circular orbits, variation of the speed of sound with temperature and of the period of a pendulum with changes in g, and the classical limits of such relativistic quantities as kinetic energy. Another important example, presented later in this paper, is approximating the electrostatic potential at large distances from a charge configuration such as a dipole. However, in algebra-based courses, the formula for the series expansion is usually pulled out of thin air. The primary goal of this article is to develop derivations of the binomial series that are simple enough to be presented in such courses. Even in a calculus-based course, where students in principle should know enough math to follow Taylor's theorem and its use in formally deriving the binomial series, a more intuitive and physics-based approach would greatly increase student understanding of and facility with the series.

A simple way to derive the binomial series to a given order is to use expansions of the exponential and logarithm functions to the same order. These latter two functions appear so frequently in the introductory curriculum (for instance, in *RC* and *LR* circuits, in thermodynamic calculations of work and heat, in radioactive decay, and in simple models of drag) that it is worth the small extra time spent initially discussing the properties and expansions of these two functions before tackling the binomial series. Furthermore, not only is the binomial series less familiar to most students than the exponential and logarithm functions, it has the added complexity of depending on two independent variables (n and x, here taken to range over the real numbers) rather than only on x.

Often it suffices for a given application to approximate the binomial series $(1+x)^n$ to first order, *i.e.*, by the sum $1+nx$ of its constant and linear terms only. In this case, the derivations are so simple that it is instructive to start with them. Rarely (if ever) is it necessary in introductory physics to go beyond second order (in which a quadratic term in x is added to the series expansions). To obtain the second-order expansion, a different approach, motivated by the familiar kinematics expression for the position of an accelerating particle as a function of time, can be used to derive the results by directly connecting the math with the physics they are concurrently learning.

II. FIRST-ORDER TAYLOR EXPANSIONS

A. The Exponential Function

Consider the function $y(x)=e^x$. It is defined by two properties. First, it must pass[1] through the point $(0,1)$. Secondly, the slope of its graph at those coordinates must equal its y-value at that point, namely 1. Consequently a plot of the exponential function to first order is a line with slope $m=1$ and y-intercept $b=1$, so that

$$y=b+mx \quad \Rightarrow \quad \boxed{e^x \approx 1+x}, \tag{1}$$

is the first-order approximation of an exponential, valid for values of x near zero, *i.e.*, for $|x| << 1$.

B. The Natural Logarithm

The function $\ln(x)$ is undefined at the origin and consequently we cannot expand it in a Maclaurin series about that point. The simplest alternative is to shift the argument by a unit step and instead develop the expansion of $y(x)=\ln(1+x)$. Noting that the logarithm is the inverse of the exponential function, we can take the log of both sides of Eq. (1) written as $1+x\approx e^x$ to immediately get

$$\boxed{\ln(1+x)\approx x}, \tag{2}$$

the expansion of logarithm to first order, which again holds for values of x such that $|x|<<1$. Graphically, Eqs. (1) and (2) can be interpreted as the best linear fits to an exponential and a logarithm near $x=0$, as graphed in Fig. 1.

C. The Binomial Series

As a preliminary step, replace x by nx in Eq. (1) to obtain

$$e^{nx}\approx 1+nx\,, \tag{3}$$

which is valid provided $|nx|<<1$. It is now straightforward to obtain the binomial expansion to first order as follows,

$$(1+x)^n = e^{n\ln(1+x)} \approx e^{nx}\,, \tag{4}$$

[1]The general solution of the differential equation $dy/dx=y$ is $y(x)=Ae^x$ which passes through the point $(0,A)$ where A can be any real value.

using Eq. (2) in the second step. Now substitute Eq. (3) to get the final result,

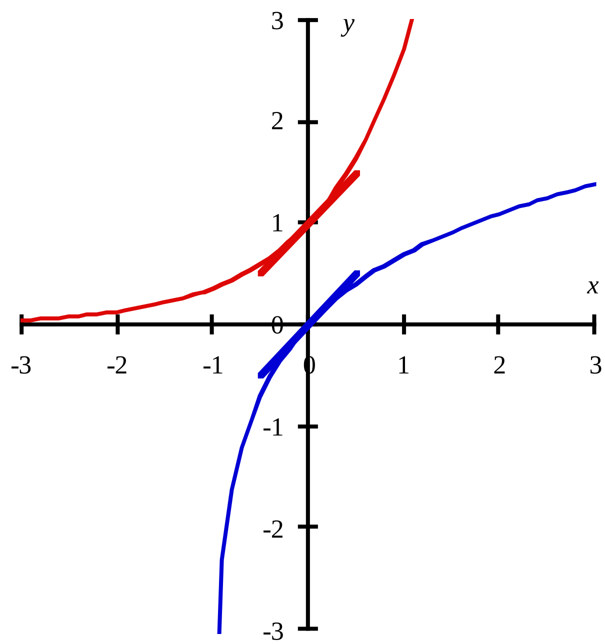

FIGURE 1. The function $y=e^x$ (thin red curve) can be approximated by $1+x$ (red line segment) for $|x|<<1$, while $y=\ln(1+x)$ plotted by the thin blue curve can be approximated by x (blue line segment) near the origin.

$$\boxed{(1+x)^n \approx 1+nx}\,. \tag{5}$$

As a quick demonstration to convince students of its validity and power, use this result to calculate $\sqrt{1.21}\approx 1+0.21/2=1.105$, in good agreement with the exact value of 1.100.

Note that Eq. (5) is only valid provided that *both* $|x|<<1$ and $|nx|<<1$. Many textbooks and instructors omit to mention the second limiting condition! As a counter-example when the first but not the second limit holds, try $x=0.1$ and $n=100$. On the other hand, try $x=20$ and $n=0.01$ for a counter-example when only the second limit holds. Both conditions are therefore *individually* necessary. The curious instructor can understand this conclusion by examining the second-order term in the binomial expansion, derived in Eq. (15) below, which is the sum of $n^2x^2/2$ and $-nx^2/2$. If these expressions are each to be negligible compared to the first-order term nx, then we require that

$$\frac{|n^2x^2|}{|nx|}<<1 \quad \text{and} \quad \frac{|nx^2|}{|nx|}<<1, \tag{6}$$

which reduce to the preceding two limiting conditions $|nx|<<1$ and $|x|<<1$. It is left as an exercise to the reader to show that these two conditions guarantee that the third and higher order terms in the binomial expansion are also negligible.

III. SECOND-ORDER TAYLOR EXPANSIONS

A. The Exponential Function

Instead of writing the independent variable as *x*, we will now write it as *t* and discuss the expansion of $y(t)=e^t$. Thinking of *y* as the position of a particle moving in one dimension during a time interval *t*, we can invoke the standard kinematics expression[2]

$$y = y_0 + \upsilon_0 t + \tfrac{1}{2} a_0 t^2 , \tag{7}$$

where y_0, υ_0, and a_0 are the position, velocity, and acceleration of the particle at $t=0$. Here a_0 has been written in place of the more familiar form *a* because the acceleration is not constant for a particle whose position varies exponentially with time. Therefore Eq. (7) is only valid for small times, that is for $|t| << 1$.

Evaluating $y(t)=e^t$ at $t=0$ gives

$$y_0 = e^0 = 1 , \tag{8a}$$

in accord with the first property of the exponential function discussed in Sec. II.A above. A more complete statement [3] of the second property of an exponential is that the slope of the function is equal to the value of the function *y* at any value of the independent variable *t*. (This property is expressed as a differential equation for independent variable *x* in Footnote 1.) Since the slope of a graph of the position versus time is the velocity, it follows that $\upsilon(t) = y(t) = e^t$ and therefore

$$\upsilon_0 = 1 . \tag{8b}$$

Similarly, the slope of the velocity defines the acceleration, so that $a(t)=e^t$ also, and thus

$$a_0 = 1 . \tag{8c}$$

Substituting Eqs. (8a) to (8c) into (7) along with $y(t)=e^t$ finally gives

$$\boxed{e^t = 1 + t + \tfrac{1}{2} t^2} , \tag{9}$$

which is an expansion of the exponential function up to the second power of its argument *t*. As expected from Fig. 1, the coefficient of the second-order term is positive because the plot of the exponential rises above the red first-order line segment at both of its ends.

[2]Implicitly, both *y* and *t* are assumed to be dimensionless by having normalized them to some characteristic length and time scales.

B. The Natural Logarithm

Ignoring a term of order t^3, Eq. (9) can be rewritten as

$$e^t = (1+t)\left(1+\tfrac{1}{2}t^2\right) . \tag{10}$$

Taking the logarithm of both sides gives

$$t = \ln(1+t) + \ln\left(1+\tfrac{1}{2}t^2\right) \approx \ln(1+t) + \tfrac{1}{2}t^2 , \tag{11}$$

using Eq. (2) to expand the second logarithm to first order in t^2, so that Eq. (11) is valid up to the second power of *t*. It rearranges into

$$\boxed{\ln(1+t) \approx t - \tfrac{1}{2}t^2} , \tag{12}$$

which is the desired second-order expansion of the logarithm, valid for $|t| << 1$. This time the coefficient of the second-order term is negative, in accord with the fact that the graph of the logarithm in Fig. 1 falls below its first-order linear approximation (represented by the blue line segment) as one moves away from the origin in either direction.

C. The Binomial Series

Proceeding in analogy to Sec. II.C above, write

$$(1+x)^n = e^{n\ln(1+x)} \approx e^{n\left(x-\frac{1}{2}x^2\right)} , \tag{13}$$

using Eq. (12) in the second step with *x* written in place of *t*. Then equate the right-hand sides of Eqs. (9) and (13) with the argument of the final exponential in Eq. (13) replacing *t* in Eq. (9) to get

$$(1+x)^n \approx 1 + n\left(x - \tfrac{1}{2}x^2\right) + \tfrac{1}{2}n^2\left(x-\tfrac{1}{2}x^2\right)^2 . \tag{14}$$

Finally expand the last square and discard all terms with powers of *x* larger than 2 to obtain

$$\boxed{(1+x)^n \approx 1 + nx + \frac{n(n-1)}{2}x^2} . \tag{15}$$

IV. EXAMPLE OF A FIRST-ORDER BINOMIAL EXPANSION IN ELECTROSTATICS

Given point charges $\pm Q$ separated by distance *a* in Fig. 2, find the electrostatic potential *V* at the starred location assuming $y >> a$.

Denoting the Coulomb constant as *k*, the potential (with the reference at infinity as usual) is the sum of that due to each point charge alone,

$$V=\frac{kQ}{y}-\frac{kQ}{\sqrt{y^2+a^2}}=\frac{kQ}{y}\left[1-\left(1+\frac{a^2}{y^2}\right)^{-1/2}\right],$$
$$\approx\frac{kQ}{y}\left[1-\left(1-\frac{1}{2}\frac{a^2}{y^2}\right)\right]=\boxed{\frac{kQa^2}{2y^3}}, \qquad (16)$$

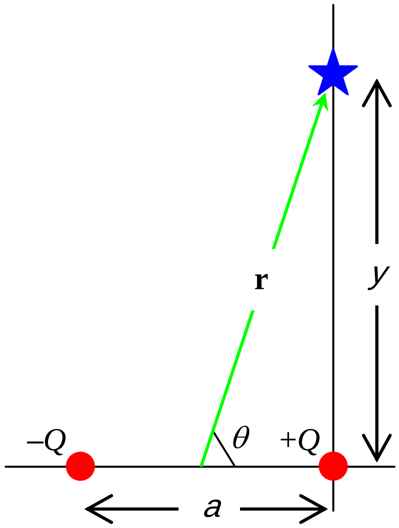

FIGURE 2. The electrostatic potential is to be found at the starred position located far along a perpendicular drawn from the positive end of an electric dipole.

to lowest nonzero order, using Eq. (5). Notice that the first-order term in the binomial expansion is required to obtain this result. In contrast, if one calculates the potential at some large distance x to the right of the positive charge, the answer is $V=kQa/x(x+a)\approx kQa/x^2$ which can be obtained by simply dropping a compared to x in the parentheses, equivalent to retaining only the trivial zeroth-order term $(1+z)^n\approx 1$ in a binomial expansion.

The final boxed answer in Eq. (16) for the potential does *not* fall off from the electric dipole with the inverse distance squared as $y\to\infty$. The general formula [4] for V in the far field is $kQa\cos\theta/r^2$ where $\mathbf{r}$ is the position of the field point relative to the center of the dipole (as indicated in Fig. 2) and θ is the angle between vector $\mathbf{r}$ and the dipole moment (which points horizontally to the right in the present case). We see from Fig. 2 that $\cos\theta=a/2r$ so that V falls off as the inverse distance cubed. The angle θ approaches 90° as $y\to\infty$ so that $\cos\theta$ approaches zero and V falls to zero faster than it would if θ were constant. The potential for a dipole only falls off with the inverse distance squared if one recedes *along a purely radial path* away from the midpoint between the centers of positive and negative charge. That happens for example for the case of large x discussed above (corresponding to $\theta=0$).

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